Solução da Lista 3 (by Orengo)

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(1) (a) $\vec{\nabla}\cdot\vec{\nabla}\varphi = \nabla^2\varphi$; com $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

temos $\vec{\nabla}\varphi = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}$ $\boxed{1}$

e:

$\hat{\imath}\cdot\hat{\imath} = 1$ $\hat{\imath}\cdot\hat{\jmath} = 0$ $\hat{\imath}\cdot\hat{k} = 0$

$$\left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}\right)$$

$$= \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = \nabla^2\varphi$$

Assim: $\boxed{\vec{\nabla}\cdot\vec{\nabla}\varphi = \nabla^2\varphi}$ $\boxed{2}$ e será nulo se φ for o potencial elétrico, na ausência de carga prescrita.

(b) $\vec{\nabla}\times\vec{\nabla}\varphi = 0$

utilizando $\boxed{1}$, temos:

$$\vec{\nabla}\times\left(\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}\right) = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)\times\left(\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}\right)$$

$$= \frac{\partial^2\varphi}{\partial x\partial y}\hat{k} - \frac{\partial^2\varphi}{\partial x\partial z}\hat{\jmath} - \frac{\partial^2\varphi}{\partial y\partial x}\hat{k} + \frac{\partial^2\varphi}{\partial y\partial z}\hat{\imath} + \frac{\partial^2\varphi}{\partial z\partial x}\hat{\jmath} - \frac{\partial^2\varphi}{\partial z\partial y}\hat{\imath}$$

$= 0$, logo: $\boxed{\vec{\nabla}\times\vec{\nabla}\varphi = 0}$

(c) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{V}) = 0$

temos:
$$\vec{\nabla}\times\vec{V} = \underbrace{\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{k}}_{\boxed{3}}$$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{V}) = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)\cdot(\quad)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

$$= \cancel{\frac{\partial^2 V_z}{\partial x\partial y}} - \cancel{\frac{\partial^2 V_y}{\partial x\partial z}} + \frac{\partial^2 V_x}{\partial y\partial z} - \cancel{\frac{\partial^2 V_z}{\partial y\partial x}} + \cancel{\frac{\partial^2 V_y}{\partial z\partial x}} - \frac{\partial^2 V_x}{\partial z\partial y} = 0$$

logo: $\boxed{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{V}) = 0}$

(d) $\vec{\nabla}\times(\vec{\nabla}\times\vec{V}) = \vec{\nabla}\vec{\nabla}\cdot\vec{V} - \vec{\nabla}\cdot\vec{\nabla}\vec{V}$; com $\vec{V} = V_x\hat{\imath} + V_y\hat{\jmath} + V_z\hat{k}$

$$\underbrace{\vec{\nabla}}_{\left[\frac{\partial}{\partial x}\hat{\imath}+\frac{\partial}{\partial y}\hat{\jmath}+\frac{\partial}{\partial z}\hat{k}\right]}\times\left[\left(\frac{\partial V_z}{\partial y}-\frac{\partial V_y}{\partial z}\right)\hat{\imath}+\left(\frac{\partial V_x}{\partial z}-\frac{\partial V_z}{\partial x}\right)\hat{\jmath}+\left(\frac{\partial V_y}{\partial x}-\frac{\partial V_x}{\partial y}\right)\hat{k}\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial V_x}{\partial z}-\frac{\partial V_z}{\partial x}\right)\hat{k} - \frac{\partial}{\partial x}\left(\frac{\partial V_y}{\partial x}-\frac{\partial V_x}{\partial y}\right)\hat{\jmath}$$

$$- \frac{\partial}{\partial y}\left(\frac{\partial V_z}{\partial y}-\frac{\partial V_y}{\partial z}\right)\hat{k} + \frac{\partial}{\partial y}\left(\frac{\partial V_y}{\partial x}-\frac{\partial V_x}{\partial y}\right)\hat{\imath}$$

$$+ \frac{\partial}{\partial z}\left(\frac{\partial V_z}{\partial y}-\frac{\partial V_y}{\partial z}\right)\hat{\jmath} - \frac{\partial}{\partial z}\left(\frac{\partial V_x}{\partial z}-\frac{\partial V_z}{\partial x}\right)\hat{\imath}$$

$$= \frac{\partial^2 V_x}{\partial x\partial z}\hat{k}\boxed{-\frac{\partial^2 V_z}{\partial x^2}\hat{k}}\boxed{-\frac{\partial^2 V_y}{\partial x^2}\hat{\jmath}}+\frac{\partial^2 V_x}{\partial x\partial y}\hat{\jmath}$$

$$\boxed{-\frac{\partial^2 V_z}{\partial y^2}\hat{k}}+\frac{\partial^2 V_y}{\partial y\partial z}\hat{k}+\frac{\partial^2 V_y}{\partial y\partial x}\hat{\imath}\boxed{-\frac{\partial^2 V_x}{\partial y^2}\hat{\imath}}$$

$$+\frac{\partial^2 V_z}{\partial z\partial y}\hat{\jmath}\boxed{-\frac{\partial^2 V_y}{\partial z^2}\hat{\jmath}}\boxed{-\frac{\partial^2 V_x}{\partial z^2}\hat{\imath}}+\frac{\partial^2 V_z}{\partial z\partial x}\hat{\imath}$$

SOMANDO E SUBTRAINDO A EXPRESSÃO

$$\frac{\partial^2 V_x}{\partial x^2}\hat{\imath}+\frac{\partial^2 V_y}{\partial y^2}\hat{\jmath}+\frac{\partial^2 V_z}{\partial z^2}\hat{k}$$

TEREMOS → NA OUTRA FOLHA

NOTA: ESTA CONCLUSÃO DE ADICIONAR/SUBTRAIR TERMOS NÃO SAIU DO "NADA", MAS SIM DE UM PERÍODO (LONGO) DE ANÁLISE DA SITUAÇÃO. (Por isso é necessário estudar diariamente)

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{V}) =$$

$$\frac{\partial^2 V_x}{\partial x\partial z}\hat{k} \boxed{-\frac{\partial^2 V_z}{\partial x^2}\hat{k}} \boxed{-\frac{\partial^2 V_y}{\partial x^2}\hat{j}} + \frac{\partial^2 V_x}{\partial x\partial y}\hat{j} \boxed{-\frac{\partial^2 V_z}{\partial y^2}\hat{k}} + \frac{\partial^2 V_y}{\partial y\partial z}\hat{k}$$

$$+ \frac{\partial^2 V_y}{\partial y\partial x}\hat{\imath} \boxed{-\frac{\partial^2 V_x}{\partial y^2}\hat{\imath}} + \frac{\partial^2 V_z}{\partial z\partial y}\hat{j} \boxed{-\frac{\partial^2 V_y}{\partial z^2}\hat{j}} \boxed{-\frac{\partial^2 V_x}{\partial z^2}\hat{\imath}} + \frac{\partial^2 V_z}{\partial z\partial x}\hat{\imath}$$

$$+ \frac{\partial^2 V_x}{\partial x^2}\hat{\imath} + \frac{\partial^2 V_y}{\partial y^2}\hat{j} + \frac{\partial^2 V_z}{\partial z^2}\hat{k} \boxed{-\frac{\partial^2 V_x}{\partial x^2}\hat{\imath}} \boxed{-\frac{\partial^2 V_y}{\partial y^2}\hat{j}} \boxed{-\frac{\partial^2 V_z}{\partial z^2}\hat{k}}$$

REARRANJANDO OS TERMOS, E ALTERANDO A ORDEM DE ALGUMAS DERIVADAS:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{V}) =$$

$$\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_y}{\partial x\partial y} + \frac{\partial^2 V_z}{\partial x\partial z}\right)\hat{\imath} + \left(\frac{\partial^2 V_x}{\partial y\partial x} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_z}{\partial y\partial z}\right)\hat{j}$$

$$+ \left(\frac{\partial^2 V_x}{\partial z\partial x} + \frac{\partial^2 V_y}{\partial z\partial y} + \frac{\partial^2 V_z}{\partial z^2}\right)\hat{k}$$

$$-\left(\frac{\partial^2 V_x}{\partial x^2}\hat{\imath} + \frac{\partial^2 V_y}{\partial x^2}\hat{j} + \frac{\partial^2 V_z}{\partial x^2}\hat{k} + \frac{\partial^2 V_x}{\partial y^2}\hat{\imath} + \frac{\partial^2 V_y}{\partial y^2}\hat{j} + \frac{\partial^2 V_z}{\partial y^2}\hat{k}\right.$$

$$\left. + \frac{\partial^2 V_x}{\partial z^2}\hat{\imath} + \frac{\partial^2 V_y}{\partial z^2}\hat{j} + \frac{\partial^2 V_z}{\partial z^2}\hat{k}\right)$$

SEGUE QUE:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{V}) = \frac{\partial}{\partial x}\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)\hat{\imath} + \frac{\partial}{\partial y}\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)\hat{j}$$

$$+ \frac{\partial}{\partial z}\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)\hat{k}$$

$$-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(V_x\hat{\imath} + V_y\hat{j} + V_z\hat{k}\right)$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{V}) = \overbrace{\left(\frac{\partial}{\partial x}\hat{\imath}+\frac{\partial}{\partial y}\hat{\jmath}+\frac{\partial}{\partial z}\hat{k}\right)}^{\vec{\nabla}}\overbrace{\left(\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}\right)}^{\vec{\nabla}\cdot\vec{V}}$$

$$-\underbrace{\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)}\underbrace{\left(V_x\hat{\imath}+V_y\hat{\jmath}+V_z\hat{k}\right)}_{\vec{V}}$$

$$\underbrace{\left(\frac{\partial}{\partial x}\hat{\imath}+\frac{\partial}{\partial y}\hat{\jmath}+\frac{\partial}{\partial z}\hat{k}\right)}_{\vec{\nabla}}\cdot\underbrace{\left(\frac{\partial}{\partial x}\hat{\imath}+\frac{\partial}{\partial y}\hat{\jmath}+\frac{\partial}{\partial z}\hat{k}\right)}_{\vec{\nabla}}$$

ASSIM:

$$\boxed{\vec{\nabla}\times(\vec{\nabla}\times\vec{V}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{V}) - \vec{\nabla}\cdot\vec{\nabla}\vec{V}}$$

OU, UTILIZANDO O RESULTADO DO ITEM (d), DA EQ. $\boxed{2}$:

$$\vec{\nabla}\cdot\vec{\nabla} = \nabla^2$$

$$\boxed{\vec{\nabla}\times(\vec{\nabla}\times\vec{V}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{V}) - \nabla^2\vec{V}}$$

② DADAS AS EQ. DE MAXWELL, NO VÁCUO:

$$\vec{\nabla}\cdot\vec{E}=0 \qquad (1)$$

$$\vec{\nabla}\cdot\vec{B}=0 \qquad (2)$$

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \qquad (3)$$

$$\vec{\nabla}\times\vec{B}=\varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} \qquad (4)$$

(a) APLICANDO O OPERADOR $\vec{\nabla}\times$ NA EQ. (3):

$$\underbrace{\vec{\nabla}\times\vec{\nabla}\times\vec{E}}_{\text{DO ITEM (d) - QUESTÃO 1}}=\vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right)$$ e trocando a ordem das derivações

$$\vec{\nabla}\underbrace{\vec{\nabla}\cdot\vec{E}}_{\text{DA EQ. (1)}}-\vec{\nabla}\cdot\vec{\nabla}\vec{E}=-\frac{\partial}{\partial t}\underbrace{(\vec{\nabla}\times\vec{B})}_{\text{DA EQ. (4)}}$$

LOGO: $$-\vec{\nabla}\cdot\vec{\nabla}\vec{E}=-\frac{\partial}{\partial t}\left(\varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}\right)$$

OU $$\boxed{\vec{\nabla}\cdot\vec{\nabla}\vec{E}-\varepsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2}=0}$$ OU $$\boxed{\nabla^2\vec{E}-\varepsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2}=0}$$

(b) PARA OBTER A MESMA EQ. PARA O $\vec{B}$, FAÇAMOS $\vec{\nabla}\times$ NA EQ. (4):

$$\underbrace{\vec{\nabla}\times\vec{\nabla}\times\vec{B}}_{\text{DO ITEM (d) - QUESTÃO 1}}=\vec{\nabla}\times\left(\varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}\right)$$, trocando a ordem das derivações

$$\vec{\nabla}\underbrace{\vec{\nabla}\cdot\vec{B}}_{\text{DA EQ. (2)}}-\vec{\nabla}\cdot\vec{\nabla}\vec{B}=\varepsilon_0\mu_0\frac{\partial}{\partial t}\underbrace{(\vec{\nabla}\times\vec{E})}_{\text{DA EQ. (3)}}$$

ASSIM: $$-\vec{\nabla}\cdot\vec{\nabla}\vec{B}=-\varepsilon_0\mu_0\frac{\partial^2\vec{B}}{\partial t^2}$$

OU $$\boxed{\vec{\nabla}\cdot\vec{\nabla}\vec{B}-\varepsilon_0\mu_0\frac{\partial^2\vec{B}}{\partial t^2}=0}$$ OU $$\boxed{\nabla^2\vec{B}-\varepsilon_0\mu_0\frac{\partial^2\vec{B}}{\partial t^2}=0}$$

(c) COMPARANDO A EQ. (5) COM UMA EQUAÇÃO DE ONDA MECÂNICA CONVENCIONAL:

$$\frac{\partial^2 y}{\partial x^2} - \left(\frac{1}{v^2}\right)\frac{\partial^2 y}{\partial t^2} = 0$$

⇒ DO LIVRO
Física, Volume 2 - 3ª ed. 1994
Paul Tipler - Ed. Guanabara Koogan
PÁG. 128

$$\nabla^2 \vec{E} - (\varepsilon_0 \mu_0)\frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

⇒ EQ. (5)

DEPENDÊNCIA ESPACIAL

DEPENDÊNCIA TEMPORAL

$$\frac{1}{v^2} = \varepsilon_0 \mu_0$$

OU

$$v = \sqrt{\frac{1}{\varepsilon_0 \mu_0}} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

Os valores de ε_0 e μ_0:

$\varepsilon_0 = 8{,}854 \times 10^{-12}\ C^2/N \cdot m^2$ ⇒ pág. 38

$\mu_0 = 4\pi \times 10^{-7}\ Ns^2/C^2$ ⇒ pág. 162

LIVRO:
FUNDAMENTOS DA TEORIA ELETROMAGNÉTICA.
Reitz, J.R., Milford, F.J., Christy, R.W.
3ª ed. 1988
Ed. Campus

Assim, teremos:

$$v = \frac{1}{\sqrt{8{,}854 \times 10^{-12}\ \frac{C^2}{Nm^2}\ 4\pi \times 10^{-7}\ \frac{Ns^2}{C^2}}} = 2{,}997956\ m/s$$

CONCLUSÃO: $v = c$; A VELOCIDADE DA LUZ NO VÁCUO.

③ $\vec{\nabla}\cdot(u\vec{\nabla}v) = u\vec{\nabla}\cdot\vec{\nabla}v + (\vec{\nabla}u)\cdot(\vec{\nabla}v)$

Então; da definição de divergente

$$\vec{\nabla}\cdot(u\vec{\nabla}v) = \frac{\partial}{\partial x}\underbrace{(u\vec{\nabla}v)_x}_{\text{parte em } x,\ \frac{\partial v}{\partial x}} + \frac{\partial}{\partial y}(u\vec{\nabla}v)_y + \frac{\partial}{\partial z}(u\vec{\nabla}v)_y$$

$$= \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(u\frac{\partial v}{\partial z}\right)$$

$$= u\frac{\partial^2 v}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial y^2} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}$$

$$+ u\frac{\partial^2 v}{\partial z^2} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z}$$

REARRANJANDO OS TERMOS:

$$\vec{\nabla}\cdot(u\vec{\nabla}v) = \underbrace{u\frac{\partial^2 v}{\partial x^2} + u\frac{\partial^2 v}{\partial y^2} + u\frac{\partial^2 v}{\partial z^2}}_{u\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)} + \underbrace{\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z}}_{\text{é o mesmo que } (\vec{\nabla}u)\cdot(\vec{\nabla}v)}$$

$$u\underbrace{\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)}_{\text{DO ITEM (a), QUESTÃO ①: } \vec{\nabla}\cdot\vec{\nabla}v}$$

$$\boxed{\vec{\nabla}\cdot(u\vec{\nabla}v) = u\vec{\nabla}\cdot\vec{\nabla}v + (\vec{\nabla}u)\cdot(\vec{\nabla}v)}$$

④ SE $\vec{\nabla}\times\vec{F} = \vec{\nabla}\times\vec{G}$, então

(a) $\vec{F} = \vec{G} + K \Rightarrow \vec{\nabla}\times\vec{F} = \vec{\nabla}\times(\vec{G}+K)$

$$= \vec{\nabla}\times\vec{G} + \underbrace{\vec{\nabla}\times K}_{\text{é nulo}}$$

$$\underline{\vec{\nabla}\times\vec{F} = \vec{\nabla}\times\vec{G}}\ \checkmark$$

(b) $\vec{F} = \vec{G} + \vec{\nabla}\varphi \Rightarrow \vec{\nabla}\times\vec{F} = \vec{\nabla}\times(\vec{G}+\vec{\nabla}\varphi)$

$$= \vec{\nabla}\times\vec{G} + \underbrace{\vec{\nabla}\times\vec{\nabla}\varphi}_{\text{do item (b), questão ① (é nulo)}}$$

$$\underline{\vec{\nabla}\times\vec{F} = \vec{\nabla}\times\vec{G}}\ \checkmark$$