GABARITO - PROVA 1 (2)/2009

By ΦRENGO

(1) PARTINDO DE:

$$\sum_i (r_i)^2 = \sum_i (r_i')^2 \quad \text{E USANDO} \quad r_i' = \sum_j a_{ij} r_j$$

TEMOS

$$\sum_i (r_i')^2 = \sum_i \Big(\sum_j a_{ij} r_j\Big)^2 = \sum_i \left[\Big(\sum_j a_{ij} r_j\Big)\Big(\sum_k a_{ik} r_k\Big)\right]$$

$$= \sum_j \sum_k \boxed{\sum_i a_{ij} a_{ik}}\, r_j r_k$$

$$\delta_{jk} = \begin{cases} 1, & \text{SE } j=k \\ 0, & \text{SE } j \neq k \end{cases}$$

$$= \sum_j \sum_k \delta_{jk} r_j r_k$$

NOS SOMATÓRIOS SÓ RESTARÃO OS CASOS EM $j=k$, LOGO

$$\sum_i (r_i')^2 = \sum_j (r_j)^2, \quad \text{COMO } i=1,2,3 \text{ e } j=1,2,3, \text{ ENTÃO}$$

$$\boxed{\sum_i (r_i')^2 = \sum_i (r_i)^2}$$

CONCLUSÃO, O COMPRIMENTO DO VETOR $\vec{r} = r_x\hat{\imath} + r_y\hat{\jmath} + r_z\hat{k}$ FICA INVARIANTE A UMA ROTAÇÃO DOS EIXOS COORDENADOS.

Lembrete: os índices $i = 1,2,3 = j$ se referem

a $j=i=1 \to x$

$j=i=2 \to y$

$j=i=3 \to z$

(2) (a) VERIFICANDO $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

$$\vec{A}\times\underline{(\vec{B}\times\vec{C})} = \vec{A}\times\Big[(B_yC_z - B_zC_y)\hat{\imath} + (B_zC_x - B_xC_z)\hat{\jmath} + (B_xC_y - B_yC_x)\hat{k}\Big]$$

Por determinante, para tornar mais evidente e claro, temos:

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_yC_z - B_zC_y & B_zC_x - B_xC_z & B_xC_y - B_yC_x \end{vmatrix}$$

$$= A_yB_xC_y\hat{\imath} - A_yB_yC_x\hat{\imath} + A_zB_yC_z\hat{\jmath} - A_zB_zC_y\hat{\jmath}$$
$$+ A_xB_zC_x\hat{k} - A_xB_xC_z\hat{k} - A_zB_zC_x\hat{\imath} + A_zB_xC_z\hat{\imath}$$
$$- A_xB_xC_y\hat{\jmath} + A_xB_yC_x\hat{\jmath} - A_yB_yC_z\hat{k} + A_yB_zC_y\hat{k}$$

AGRUPANDO OS TERMOS, EM ACORDO COM A EXPRESSÃO DESEJADA, TEMOS:

$$\vec{A}\times(\vec{B}\times\vec{C})$$
$$= B_x\hat{\imath}(A_yC_y + A_zC_z) + B_y\hat{\jmath}(A_xC_x + A_zC_z) + B_z\hat{k}(A_xC_x + A_yC_y)$$
$$- \Big[C_x\hat{\imath}(A_yB_y + A_zB_z) + C_y\hat{\jmath}(A_xB_x + A_zB_z) + C_z\hat{k}(A_xB_x + A_yB_y)\Big]$$

Observando, verificamos que faltam termos nas expressões entre parenteses para completarem o produto escalar. Assim, adicionaremos os termos necessários, ao mesmo tempo subtrairemos os mesmos. Ou seja

$$B_x\hat{\imath}\,A_xC_x + B_y\hat{\jmath}\,A_yC_y + B_z\hat{k}\,A_zC_z$$
$$- C_x\hat{\imath}\,A_xB_x - C_y\hat{\jmath}\,A_yB_y - C_z\hat{k}\,A_zB_z \Rightarrow$$ percebа que estes completam os termos entre os colchetes

Teremos, então:

$$\vec{A}\times(\vec{B}\times\vec{C})$$

$$= B_x\hat{i}\,(A_xC_x + A_yC_y + A_zC_z) + B_y\hat{j}\,(A_xC_x + A_yC_y + A_zC_z)$$

$$+ B_z\hat{k}\,(A_xC_x + A_yC_y + A_zC_z)$$

$$- \Big[C_x\hat{i}\,(A_xB_x + A_yB_y + A_zB_z) + C_y\hat{j}\,(A_xB_x + A_yB_y + A_zB_z)$$

$$+ C_z\hat{k}\,(A_xB_x + A_yB_y + A_zB_z) \Big]$$

$$= \underbrace{(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})}_{\vec{B}}\underbrace{(A_xC_x + A_yC_y + A_zC_z)}_{\vec{A}\cdot\vec{C}} - \underbrace{(C_x\hat{i} + C_y\hat{j} + C_z\hat{k})}_{\vec{C}}\underbrace{(A_xB_x + A_yB_y + A_zB_z)}_{\vec{A}\cdot\vec{B}}$$

$$\boxed{\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})} \qquad (1)$$

(b) $(\vec{A}\cdot\vec{B})^2 - \Big[\underbrace{(\vec{A}\times\vec{B})\times\vec{B}}\Big]\cdot\vec{A}$

$$-\underbrace{\underset{\vec a}{\vec{B}}\times(\underset{\vec b}{\vec{A}}\times\underset{\vec c}{\vec{B}})}$$ $\Longrightarrow$ p/ usar a eq. (1).

$\vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$ $\Longrightarrow$ usado para não errar!!!

$$\underbrace{\vec{A}(\vec{B}\cdot\vec{B}) - \vec{B}(\vec{B}\cdot\vec{A})}$$

só atua entre vetores

$$= (\vec{A}\cdot\vec{B})^2 + \Big[\vec{A}(\underbrace{\vec{B}\cdot\vec{B}}_{\text{escalar}}) - \vec{B}(\underbrace{\vec{B}\cdot\vec{A}}_{\text{escalar}})\Big]\cdot\vec{A}$$

como $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$

$$= (\vec{A}\cdot\vec{B})^2 + \underbrace{\vec{A}\cdot\vec{A}}_{A^2}(\underbrace{\vec{B}\cdot\vec{B}}_{B^2}) - \vec{A}\cdot\vec{B}\,(\vec{A}\cdot\vec{B})$$

$$= (\vec{A}\cdot\vec{B})^2 + A^2B^2 - (\vec{A}\cdot\vec{B})^2$$

Assim:

$$\boxed{(\vec{A}\cdot\vec{B})^2 - \Big[(\vec{A}\times\vec{B})\times\vec{B}\Big]\cdot\vec{A} = A^2B^2}$$

③ $\varphi = x^2y + xz$

$\vec{A} = 2\hat{\imath} - 2\hat{\jmath} + \hat{k}$; $\vec{\nabla} = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$

(a) $\vec{\nabla}\varphi = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)(x^2y + xz)$

$$= y\,2x\,\hat{\imath} + z\hat{\imath} + x^2\hat{\jmath} + x\hat{k}$$

$$\boxed{\vec{\nabla}\varphi = (2xy+z)\hat{\imath} + x^2\hat{\jmath} + x\hat{k}}$$

(b) $\hat{u}$, na direção de $\vec{A}$: $|\vec{A}| = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{9} = 3$

$$\hat{u} = \frac{\vec{A}}{|\vec{A}|} = \frac{2\hat{\imath} - 2\hat{\jmath} + \hat{k}}{3} = \frac{2}{3}\hat{\imath} - \frac{2}{3}\hat{\jmath} + \frac{1}{3}\hat{k}$$

$$\boxed{\hat{u} = \frac{2}{3}\hat{\imath} - \frac{2}{3}\hat{\jmath} + \frac{1}{3}\hat{k}}$$

(c) $\left.\frac{d\varphi}{ds}\right|_{(1,2,-1)} = \left.\underbrace{\vec{\nabla}\varphi\cdot\hat{u}}\right|_{(1,2,-1)}$

derivada direcional no ponto (1,2,−1)

$$\underbrace{\left[(2xy+z)\hat{\imath} + x^2\hat{\jmath} + x\hat{k}\right]\cdot\left(\frac{2}{3}\hat{\imath} - \frac{2}{3}\hat{\jmath} + \frac{1}{3}\hat{k}\right)}_{\frac{4}{3}xy + \frac{2z}{3} - \frac{2}{3}x^2 + \frac{x}{3}}$$

$$\left.\frac{d\varphi}{ds}\right|_{(1,2,-1)} = \left.\left(\frac{4}{3}xy + \frac{2z}{3} - \frac{2}{3}x^2 + \frac{x}{3}\right)\right|_{(1,2,-1)}$$

$(1,2,-1) \to x\ y\ z$

$$= \frac{4}{3}\times 1\times 2 + \frac{2}{3}(-1) - \frac{2}{3}(1)^2 + \frac{1}{3}$$

$$= \frac{8}{3} - \frac{2}{3} - \frac{2}{3} + \frac{1}{3} = \frac{8-4+1}{3}$$

$$\boxed{\left.\frac{d\varphi}{ds}\right|_{(1,2,-1)} = \frac{5}{3}}$$

④ Verificar $\vec{\nabla}\times(f\vec{V}) = f(\vec{\nabla}\times\vec{V}) + (\vec{\nabla}f)\times\vec{V}$

$$\vec{\nabla}\times(f\vec{V}) = \underbrace{\vec{\nabla}\times(f\vec{V})\Big|_x}_{\text{componente } x} + \vec{\nabla}\times(f\vec{V})\Big|_y + \vec{\nabla}\times(f\vec{V})\Big|_z$$

$$= \overbrace{\frac{\partial}{\partial y}(fV_z) - \frac{\partial}{\partial z}(fV_y)}^{\text{componente } x} + \frac{\partial}{\partial z}(fV_x) - \frac{\partial}{\partial x}(fV_z)$$

$$+ \frac{\partial}{\partial x}(fV_y) - \frac{\partial}{\partial y}(fV_x)$$

$$= \frac{\partial f}{\partial y}V_z + f\frac{\partial V_z}{\partial y} - \frac{\partial f}{\partial z}V_y - f\frac{\partial V_y}{\partial z}$$

$$+ \frac{\partial f}{\partial z}V_x + f\frac{\partial V_x}{\partial z} - \frac{\partial f}{\partial x}V_z - f\frac{\partial V_z}{\partial x}$$

$$+ \frac{\partial f}{\partial x}V_y + f\frac{\partial V_y}{\partial x} - \frac{\partial f}{\partial y}V_x - f\frac{\partial V_x}{\partial y}$$

REARRANJANDO OS TERMOS EM $\hat{\imath}$, $\hat{\jmath}$ E $\hat{k}$:

$$\vec{\nabla}\times(f\vec{V}) = f\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{\imath} + f\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{\jmath} + f\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{k}$$

$$+ \left(\frac{\partial f}{\partial y}V_z - \frac{\partial f}{\partial z}V_y\right)\hat{\imath} + \left(\frac{\partial f}{\partial z}V_x - \frac{\partial f}{\partial x}V_z\right)\hat{\jmath} + \left(\frac{\partial f}{\partial x}V_y - \frac{\partial f}{\partial y}V_x\right)\hat{k}$$

$$= f\Bigg[\underbrace{\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{k}}_{\vec{\nabla}\times\vec{V}}\Bigg]$$

$$+ \underbrace{\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}}_{(\vec{\nabla}f)\times\vec{V}}$$

$$\boxed{\vec{\nabla}\times(f\vec{V}) = f\,\vec{\nabla}\times\vec{V} + (\vec{\nabla}f)\times\vec{V}}$$